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Numerical investigation of solidification processes of cylindrical ingots in a metal mould at variable technological circumstances

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Abstract

The process of the solidification of a binary alloy in a cylindrical metal mould is investigated by the method of computational experiment. An equilibrium model is used for the mushy region. Variable technological peculiarities are taken into account. Some results of the numerical experiments are presented and discussed.

1. Introduction

Creating high-strength metal alloys and obtaining from them ingots of required quality are some of the most important problems in metallurgy. To solve them successfully it is necessary to have some prior information about the main regularities of the going processes and about the magnitudes characterising them. Examining these factors is a necessary prerequisite for the efficient control of the processes of crystallization. It is well known that the properties of the ingot are mainly formed during the solidification when the alloy changes from liquid into solid state [1,2]. Nowadays the main qualitative regularities of the ingot formation are known. A lot of experimental material is gathered [3]. However, the experimental investigation is not always possible and appropriate because it leads to experiments with a lot of parameters at complicated circumstances and as a result—to great material expenditures. That is why in the detailed quantitative investigation of the obtained ingots, the computational experiment [4]—a combination of mathematical modelling and numerical methods with computer usage, is acquiring greater significance.

Diverse technological methods are used to improve the quality of the ingots. It is very difficult to assess their potential when used on simplified models. The examination of real processes by the method of computational experiment requires development of more adequate models of the processes (multi-dimensional, nonlinear) [5], construction of numerical methods of a required quality, a program implementation of these methods and calculation of many technological variants.

The purpose of this paper is the investigation of the solidification of a binary alloy in a cylindrical metal mould by the method of computational experiment. An equilibrium model [6] is used for the description of the mushy region between solid and liquid phases. The effect of the basic technological methods on the quality of ingots is studied. In particular, the gas gapping and the mould paint between ingot and metal mould are taken into account. The

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Nomenclature

Bi	Biot number
c	specific heat
d_s	thickness of ingot shell
h	space step
Ki	Kirpichev number
l_g	width of the gas gapping
l_v	thickness of the mould paint
L	latent heat of phase change
q	heat flow on the upper part of the head
Q	heat flow between the ingot and the mould
r, z	cylindrical coordinates
Ste	Stefan number
t	time
t_q	moment of complete combustion of exothermal mixtures
T, u	temperature
T_v	temperature of the inner surface of the mould paint
y	approximate solution.

Greek symbols

α_k	contact coefficient of heat transfer
α_1, α_2, α_3	heat conduction coefficients
η	temperature width of the mushy region
λ	thermal conductivity
ρ	density
σ_1, σ_2	heat radiation coefficients
τ	time step
ψ	solid phase volume fraction.

Subscripts

f	mould
g	gas gapping
k	basic part of the mould
l	liquid or liquidus
m	metal alloy
n	bush
s	solid or solidus
v	mould paint
∞	environment.

Superscript

$'$	dimensionless variables.

effect of the bush and the exothermal mixtures are examined.

2. Description of the model

2.1. Physical assumptions

We consider the processes of heat transfer during the solidification of a binary alloy in a cylindrical metal mould (Fig. 1). As it is well known the crystallization of the alloys is characterised by the existence of a two-phase zone (mushy region) between liquid and solid phase [6,7]. Variable models for the mushy region are considered [6]. An equilibrium model for the two-phase zone is assumed in this paper. In this model it is supposed that the process of solidification takes place in the temperature interval (T_s, T_l). The latent heat of phase change L is given out in this temperature interval. It is taken into account by defining the effective specific heat.

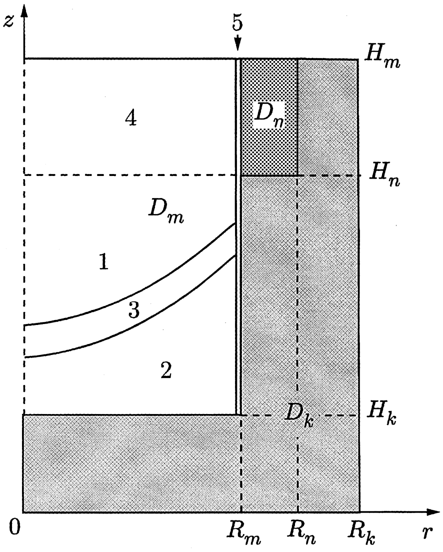

Fig. 1. D_m: 1—liquid phase; 2—solid phase; 3—mushy region; 4—head; 5—thin thermal resistance (gas gapping, mould paint). D_n—bush; D_k—basic part of the mould. $D_f = D_k \cup D_n$, $D = D_m \cup D_f$.

In metallurgy it is well known, that to obtain qualitative ingots it is necessary to realize such conditions for the head that it crystallizes last. This may be done by minimizing the heat flow via the lateral surfaces of the head and by increasing its heat content. That is why a bush of a heat-insulating material is mounted in the upper part of the metal mould and the head is heated by hypergolic exothermal mixtures. The heat which is generated during their combustion is given by equivalent heat flow q. It acts for a fixed time t_q.

There is a thin thermal resistance (gas gapping, mould paint) between the alloy and the metal mould. The heat transfer with the environment is given by heat conduction of Newton's law and by heat radiation of Stefan–Boltzmann's law type. As usual, it is assumed that the metal mould is filled instantly by an alloy and that the initial alloy temperature is a constant.

The convective flow of the melt is neglected in the considered model. Such an approach is possible in the simulation of small ingots.

We also note that the process of crystallization of cylindrical ingots in axially symmetric conditions of cooling is not dependent on the angle of the rotation, so the problem is two-dimensional.

2.2. Mathematical model

The temperature field in the region D (Fig. 1) is described by the usual heat equation

$$c\rho\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right), \quad (r,z) \in D, \quad t > 0. \tag{1}$$

For the coefficients c, ρ, λ we assume that

$$c(r,z,T), \quad \rho(r,z), \quad \lambda(r,z,T) = \begin{cases} c_m(T),\ \rho_m,\ \lambda_m(T), & (r,z) \in D_m \\ c_k,\ \rho_k,\ \lambda_k, & (r,z) \in D_k \\ c_n,\ \rho_n,\ \lambda_n, & (r,z) \in D_n \end{cases}.$$

In the context of the assumed equilibrium model of the mushy region for the specific heat in the ingot we have

$$c_m(T) = \begin{cases} c_s, & T < T_s \\ c_s - L\dfrac{d\psi(T)}{dT}, & T_s \le T \le T_l \\ c_l, & T > T_l \end{cases} \tag{2}$$

and for the heat conductivity

$$\lambda_m(T) = \begin{cases} \lambda_s, & T \le T_l \\ \lambda_l, & T > T_l \end{cases}.$$

We consider that $\psi(T)$ is a continued function of the temperature:

$$\psi(T) = \begin{cases} 1, & T \le T_s \\ \dfrac{T_l - T}{a(T_s - T) + (T_l - T_s)}, & T_s \le T \le T_l \\ 0, & T \ge T_l \end{cases}$$

where a is a parameter.

Eq. (1) is completed by the corresponding boundary, initial and conjugate conditions. On the axis of symmetry $r = 0$ we have

$$\lim_{r \to 0} r\lambda\frac{\partial T}{\partial r} = 0, \quad 0 \le z \le H_m. \tag{3}$$

The boundary conditions are

$$\lambda_k\frac{\partial T}{\partial r}\bigg|_{r=R_k} = -\alpha_1(T - T_\infty), \quad 0 \le z \le H_m \tag{4}$$

$$\lambda_k\frac{\partial T}{\partial z}\bigg|_{z=0} = \alpha_2(T - T_\infty), \quad 0 \le r \le R_k \tag{5}$$

$$\lambda\frac{\partial T}{\partial z}\bigg|_{z=H_m} = \begin{cases} q_1(t) - \alpha_3(T - T_\infty) - \sigma_1(T^4 - T_\infty^4), & 0 \le r \le R_m \\ -\alpha_3(T - T_\infty), & R_m \le r \le R_k \end{cases} \tag{6}$$

where

$$q_1(t) = \begin{cases} q, & 0 \le t \le t_q \\ 0, & t > t_q \end{cases}.$$

On the boundary between the bush and the basic part of the mould as on the bottom between the ingot and the mould the conditions of ideal contact are accepted. This gives the following conditions of conjugation:

$$\lambda_k \frac{\partial T}{\partial z}\bigg|_{z=H_n-0} = \lambda_n \frac{\partial T}{\partial z}\bigg|_{z=H_n+0}, \quad T\,|_{z=H_n-0} = T\,|_{z=H_n+0}\,, \quad R_m \le r \le R_n$$

$$\lambda_n \frac{\partial T}{\partial r}\bigg|_{r=R_n-0} = \lambda_k \frac{\partial T}{\partial r}\bigg|_{r=R_n+0}, \quad T\,|_{r=R_n-0} = T|_{r=R_n+0}, \quad H_n \le z \le H_m$$

$$\lambda_k \frac{\partial T}{\partial z}\bigg|_{z=H_k-0} = \lambda_m \frac{\partial T}{\partial z}\bigg|_{z=H_k+0}, \quad T\,|_{z=H_k-0} = T\,|_{z=H_k+0}\,, \quad 0 \le r \le R_m. \tag{7}$$

Let us consider the conjugate conditions at $r = R_m$, $H_k \leqslant z \leqslant H_m$ in detail. By neglecting the thickness of the thin thermal resistance we get the condition

$$Q = -\lambda_m \frac{\partial T}{\partial r}\bigg|_{r=R_m-0} = -\lambda \frac{\partial T}{\partial r}\bigg|_{r=R_m+0}, \quad H_k \le z \le H_m. \tag{8}$$

The discontinuity of the temperature at $r = R_m$, $H_k \leqslant z \leqslant H_m$ is determined by the contact condition on the boundary between the ingot and the mould. If on the inner surface of the metal mould there is not any mould paint and the gas gapping has not yet appeared then the conditions of non-ideal contact take place when together with (8) the equality

$$Q = \alpha_k(T\,|_{r=R_m-0} - T\,|_{r=R_m+0})$$

holds true. We consider, that via the gas gapping the heat transfer is realized by molecular heat conductivity and by heat radiation:

$$Q = \alpha_g(T\,|_{r=R_m-0} - T\,|_{r=R_m+0}) + \sigma_2(T^4\,|_{r=R_m-0} - T^4\,|_{r=R_m+0}) \tag{9}$$

where $\alpha_g = \lambda_g / l_g$.

The availability of the mould paint is simulated in the following manner [8]. Until the appearance of the gas gapping it is assumed that the contact between the alloy and the mould paint on the one hand and by the mould paint and the metal mould on the other is ideal and the distribution of the temperature in the mould paint is linear. For the density of heat flow Q via the mould paint we obtain

$$Q = \alpha_v(T\,|_{r=R_m-0} - T\,|_{r=R_m+0}), \quad \text{where } \alpha_v = \frac{\lambda_v}{l_v}. \tag{10}$$

After the appearance of the gas gapping between the ingot and the metal mould there is a double-layer contact surface. The heat transfer via gas gapping is simulated according to (9) by the condition

$$Q = \alpha_g(T\,|_{r=R_m-0} - T_v) + \sigma_2(T^4\,|_{r=R_m-0} - T_v^4). \tag{11}$$

For the mould paint by analogy with (10) we get

$$Q = \alpha_v(T_v - T\,|_{r=R_m+0}). \tag{12}$$

Equating (11) and (12), we obtain a nonlinear equation for determining the unknown temperature T_v by $T\,|_{r=R_m-0}$ and $T\,|_{r=R_m+0}$:

$$T_v = \frac{\alpha_v T\,|_{r=R_m+0} + (\alpha_g + \alpha_w) T\,|_{r=R_m-0}}{\alpha_v + \alpha_w + \alpha_g} \tag{13}$$

where $\alpha_w = \sigma_2(T^2\,|_r = R_m - 0 + T_v^2)(T\,|_r = R_m - 0 + T_v)$.

The peculiarity of the simulation of the gas gapping is not only the nonlinear dependence of Q on the temperatures $T\,|_{r=R_m-0}$, T_v and $T\,|_{r=R_m+0}$. More important is the fact that the gas gapping is not formed immediately but after the solidification of a definite part of the molten metal. It is assumed, that the gas gapping is formed after the thickness of the ingot shell d_s reaches the value d_{s_0}. Combining the conditions (8)–(13) we obtain

$$\lambda_m \frac{\partial T}{\partial r}\bigg|_{r=R_m-0} = \lambda \frac{\partial T}{\partial r}\bigg|_{r=R_m+0} = -\alpha(T\,|_{r=R_m-0} - T\,|_{r=R_m+0}), \quad H_k \le z \le H_m \tag{14}$$

where

$$\alpha = \begin{cases} \alpha_k, & l_v = 0, \quad d_s < d_{s_0} \\ \alpha_g + \tilde{\alpha}_w, & l_v = 0, \quad d_s \ge d_{s_0} \\ \alpha_v, & l_v \ne 0, \quad d_s < d_{s_0} \\ \dfrac{(\alpha_g + \alpha_w)\alpha_v}{\alpha_v + \alpha_w + \alpha_g}, & \lambda_v \ne 0, \quad d_s \ge d_{s_0} \end{cases} \tag{15}$$

and

$$\tilde{\alpha}_w = \sigma_2(T\,|_{r=R_m-0} + T\,|_{r=R_m+0})(T^2\,|_{r=R_m-0} + T^2\,|_{r=R_m+0}).$$

The initial condition for the Eq. (1) looks like

$$T(r,z,0)=\begin{cases} T_0, & (r,z)\in D_m \\ T_\infty, & (r,z)\in D_f \end{cases}. \tag{16}$$

Eq. (2) and conditions (3)–(7) and (13)–(16) give the full mathematical formulation of the problem under consideration.

We note, that similar models of crystallization of cylindrical ingots were considered in [9,10].

2.3. Dimensionless formulation of the problem

The dimensionless formulation of the problems (3)–(7) and (13)–(16) is done by the characteristics of the solid phase of the ingot. Let $\tilde{c}=c\rho$. If by c', λ', r', z', t' and u the dimensionless variables are denoted then we set $\tilde{c}=c'\tilde{c}_s$, $\lambda=\lambda'\lambda_s$, $T=uT_s$, $r=r'R_m$, $z=z'R_m$, $t=t't_s$, where $\tilde{c}_s=c_s\rho_m$, $t_s=\tilde{c}_sR_m^2/\lambda_s$.

Dimensionless formulation of Eq. (1) looks like

$$c'\frac{\partial u}{\partial t'}=\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\lambda'\frac{\partial u}{\partial r'}\right)+\frac{\partial}{\partial z'}\left(\lambda'\frac{\partial u}{\partial z'}\right), \quad (r',z')\in D', \quad t'>0. \tag{17}$$

For the coefficients $c'(r', z', u)$, $\lambda'(r', z', u)$ we obtain

$$c'(r',z',u),\quad \lambda'(r',z',u)=\begin{cases} c'_m(u),\lambda'_m(u), & (r',z')\in D'_m \\ c'_k,\lambda'_k, & (r',z')\in D'_k \\ c'_n,\lambda'_n, & (r',z')\in D'_n \end{cases}$$

where

$$c'_m(u)=\begin{cases} 1, & u<1 \\ 1-\dfrac{(a-1)\eta}{[a(1-u)+\eta]^2}\cdot Ste, & 1\le u\le 1+\eta \\ c'_1, & u>1+\eta \end{cases}$$

$$\lambda'_m(u)=\begin{cases} 1, & u\le 1+\eta \\ \lambda'_1, & u>1+\eta \end{cases}$$

$$\eta=u_l-u_s,\quad Ste=\frac{L}{c_sT_s}.$$

After the introduction of Biot numbers $Bi_i=(\alpha_iR_m/\lambda_s)$, $i=1,\ 2,\ 3,\ k,\ g,\ v$, $Bi_{\sigma_i}=(\sigma_iT^3R_m/\lambda_s)$, $i=1,\ 2$ and Kirpichev number $Ki\ (t')=(q_1(t)R_m/\lambda_sT_s)$, conditions (3)–(7) and (13)–(16) are written down in the form

$$\lim_{r'\to 0} r'\lambda'\frac{\partial u}{\partial r'}=0,\quad 0\le z'\le H'_m \tag{18}$$

$$\lambda'_k\frac{\partial u}{\partial r'}\bigg|_{r'=R'_k}=-Bi_1\,(u-u_\infty),\quad 0\le z'\le H'_m \tag{19}$$

$$\lambda'_k\frac{\partial u}{\partial z'}\bigg|_{z'=0}=Bi_2\,(u-u_\infty),\quad 0\le r'\le R'_k \tag{20}$$

$$\lambda'\frac{\partial u}{\partial r'}\bigg|_{z'=H'_m}=\begin{cases} Ki\,(t')-Bi_3\,(u-u_\infty)-Bi_{\sigma_1}\,(u^4-u_\infty^4), & 0\le r'\le 1 \\ -Bi_3\,(u-u_\infty), & 1\le r'\le R'_k \end{cases} \tag{21}$$

$$\lambda'_k\frac{\partial u}{\partial z'}\bigg|_{z'=H'_n-0}=\lambda'_n\frac{\partial u}{\partial z'}\bigg|_{z'=H'_n+0},$$
$$u\,|_{z'=H'_n-0}=u\,|_{z'=H'_n+0}\,,\quad 1\le r'\le R'_n$$

$$\lambda'_n\frac{\partial u}{\partial r'}\bigg|_{r'=R'_n-0}=\lambda'_k\frac{\partial u}{\partial r'}\bigg|_{r'=R'_n+0},$$
$$u\,|_{r'=R'_n-0}=u\,|_{r'=R'_n+0}\,,\quad H'_n\le z'\le H'_m$$

$$\lambda'_k\frac{\partial u}{\partial z'}\bigg|_{z'=H'_k-0}=\lambda'_m\frac{\partial u}{\partial z'}\bigg|_{z'=H'_k+0}, \quad u\,|_{z'=H'_k-0}=u\,|_{z'=H'_k+0}\,,\quad 0\le r'\le 1 \tag{22}$$

$$\lambda'_m\frac{\partial u}{\partial r'}\bigg|_{r'=1-0}=\lambda'\frac{\partial u}{\partial r'}\bigg|_{r'=1+0} =-Bi\,(u)(u\,|_{r'=1-0}-u\,|_{r'=1+0}),\quad H'_k\le z'\le H'_m \tag{23}$$

where

$$Bi\,(u)=\begin{cases} Bi_k, & l'_v=0,\quad d'_s<d'_{s_0} \\ Bi_g+\tilde{\alpha}'_w, & l'_v=0,\quad d'_s\ge d'_{s_0} \\ Bi_v, & l'_v\ne 0,\quad d'_s<d'_{s_0} \\ \dfrac{(Bi_g+\alpha'_w)\,Bi_v}{Bi_v+\alpha'_w+Bi_g}, & l'_v\ne 0,\quad d'_s\ge d'_{s_0} \end{cases}$$

and

$$\tilde{\alpha}'_w=Bi_{\sigma_2}\,(u\,|_{r'=1-0}+u\,|_{r'=1+0})(u^2\,|_{r'=1-0}+u^2\,|_{r'=1+0}),$$

$$\alpha'_w=Bi_{\sigma_2}\,(u\,|_{r'=1-0}+u_v)(u^2\,|_{r'=1-0}+u_v^2).$$

The initial condition takes the form

$$u(r',z',0)=u_0(r',z')=\begin{cases} \tilde{u}_0, & (r',z')\in D'_m \\ u_\infty, & (r',z')\in D'_f \end{cases}. \tag{24}$$

The main difficulties of the theoretical and the numerical analysis of the problem under consideration are a result from its peculiarities, as follows:

- nonlinear Eq. (17) in the complicated region D';
- moving discontinuities of the coefficients of the Eq. (17);
- nonlinear conjugate condition (23);
- discontinuous initial condition (24).

3. Numerical method

For the numerical solving of the problem (17)–(24) we apply an effective finite difference method. In the cylinder $\bar{Q}'=\bar{D}'\times[t'\geqslant 0]$ we construct a nonuniform grid

$$\omega=\bar{\omega}_{\hbar^r}\times\bar{\omega}_{\hbar^z}\times\omega_\tau$$

where

$$\bar{\omega}_{\hbar^r}=\Big\{r'_i=r'_{i-1}+h^r_i,\ i=2,\ 3,\ldots,\ N,\ N+1,\ldots,\ N_n,$$

$$N_n+1,\ldots,\ Mr'_1=\frac{h^r_1}{2},\ h^r_{N+1}=0,$$

$$r'_N=r'_{N+1}=1,\ r'_{N_n}=R'_n-\frac{h^r_{N_n+1}}{2}$$

$$r'_{N_n+1}=R'_n+\frac{h^r_{N_n+1}}{2},\ r'_M=R'_k\Big\}$$

$$\bar{\omega}_{\hbar^z}=\Big\{z'_p=z'_{p-1}+h^z_p,\ p=1,\ 2,\ldots,\ P_k,\ P_k+1,\ldots,$$

$$P_n,\ P_n+1,\ldots,\ Pz'_0=0,\ z'_{P_i}=H_i-\frac{h^z_{P_i+1}}{2},$$

$$z'_{P_i+1}=H'_i+\frac{h^z_{P_i+1}}{2},\ i=k,\ n,\ z'_P=H'_m\Big\}$$

$$\omega_\tau=\{t'_0=0,\ t'_j=t'_{j-1}+\tau_j,\ j=1,\ 2,\ldots\}.$$

The features of the constructed grid are the following: the grid $\bar{\omega}_{\hbar^r}$ is shifted at half grid size from the axis $r'=0$; the straight line $r'=1$ (along which the solution is discontinuous) is 'double'; on the straight lines $r'=R'_n,\ z'=H'_k,\ z'=H'_n$ (along which the solution is continuous) of the cell-centre nodes of the grid lie. The grid condenses near the interior boundaries of the domain D'. At small values of t', when the process is characterized by great gradients of the solution, we chose a small enough time-step which then increases several times.

To solve the quasilinear equation (17) at additional conditions (18)–(24) we apply a decomposition method [4]. In this method the problem (17)–(24) may be reduced to the consecutive solving of the one-dimensional heat equations

$$\frac{1}{2}c'\frac{\partial v_{(1)}}{\partial t'}=\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\lambda'\frac{\partial v_{(1)}}{\partial r'}\right),\quad t'_j<t'\leq t'_{j+1/2}\tag{25}$$

$$\frac{1}{2}c'\frac{\partial v_{(2)}}{\partial t'}=\frac{\partial}{\partial z'}\left(\lambda'\frac{\partial v_{(2)}}{\partial z'}\right),\quad t'_{j+1/2}<t'\leq t'_{j+1}\tag{26}$$

under the corresponding additional conditions. We also set

$$v_{(1)}(r',z',0)=u_0(r',z')$$

$$v_{(2)}(r',z',t'_{j+1/2})=v_{(1)}(r',z',t'_{j+1/2}).$$

Below the difference scheme is constructed by using a finite volume discretization [4] of these two equations. In this paper we use a nonimplicit linearized difference scheme, when the values of coefficients are evaluated by the values of the solution on the previous time step.

Let us denote by $y^j_{i,p}$ the approximate solution of the problem (17)–(24) in the node $(r'_i,\ z'_p,\ t'_j)\in\omega$. Then the difference approximation of Eq. (25), the conjugate conditions and the boundary conditions along r' look like:

$$\tilde{c}(y^j)\frac{y^{j+1/2}_{i,p}-y^j_{i,p}}{\tau_{j+1}}=\frac{1}{\hbar^r_i r'_i}[W^{(r)}_{i+1/2,p}-W^{(r)}_{i-1/2,p}]$$

$$i=1,\ 2,\ldots,\ M;\quad j=0,\ 1,\ldots;\quad p=0,\ 1,\ldots,\ P\tag{27}$$

where

$$W^{(r)}_{i-1/2,p}=\begin{cases}0, & i=1\\ r'_{i-1/2}a(y^j)\dfrac{y^{j+1/2}_{i,p}-y^{j+1/2}_{i-1,p}}{h^r_i}, & i=2,\ 3,\ldots\ N,\ N+2,\ldots,\ M\\ r'_N Bi(y^j)(y^{j+1/2}_{N+1,p}-y^{j+1/2}_{N,p}), & i=N+1\end{cases}$$

$$W^{(r)}_{M+1/2,p}=-R'_k\,Bi_1\,(y^{j+1/2}_{M,p}-u_\infty)$$

$$\tilde{c}(y^j)=\frac{1}{\hbar^r_i}\int_{r'_{i-1/2}}^{r'_{i+1/2}}c'(r',z'_p,\bar{u})\,\mathrm{d}r'$$

$$a(y^{j}) = \left[\frac{1}{h_i^r}\int_{r'_{i-1}}^{r'_i} \frac{\mathrm{d}r'}{\lambda'(r',z'_p,\bar{u})}\right]^{-1} \tag{28}$$

and $\bar{u}$ is the linear fulfilment of the difference function $y^j_{i,p}$ at fixed p. Eq. (25), the conjugate conditions and the boundary conditions along z' are approximated in the following manner

$$\bar{\bar{c}}(y^{j+1/2})\frac{y_{i,p}^{j+1}-y_{i,p}^{j+1/2}}{\tau_{j+1}} = \frac{1}{\hbar_1^z}[W^{(z)}_{i,p+1/2} - W^{(z)}_{i,p-1/2}]$$

$$i=0,\ 1,\ldots,\ M;\quad j=0,\ 1,\ldots;\quad p=1,\ 2,\ldots,\ P \tag{29}$$

where

$$W^{(z)}_{i,p-1/2} = \begin{cases} Bi_2\,(y_{i,1}^{j+1}-u_\infty), & p=1 \\ b_p(y^{j+1/2})\dfrac{y_{i,p}^{j+1}-y_{i,p-1}^{j+1}}{h_p^z}, & p=2,\ 3,\ldots,\ P \end{cases}$$

$$W^{(z)}_{i,P+1/2} = Ki\,(t_{j+1/2}) - [Bi_3 + Bi_{\sigma_2}\,((y_{i,P}^{j+1/2})^2 + u_\infty^2)(y_{i,P}^{j+1/2}+u_\infty)](y_{i,P}^{j+1}-u_\infty),\quad 0 \le r' \le 1,$$

$$-Bi_3(y_{i,P}^{j+1}-u_\infty),\quad 1 \le r' \le R'_k$$

$$\bar{\bar{c}}(y^{j+1/2}) = \frac{1}{\hbar_p^z}\int_{z'_{p-1/2}}^{z'_{p+1/2}} c'(r'_i,z'\bar{\bar{u}})\,\mathrm{d}z'$$

$$b_p(y^{j+1/2}) = \left[\frac{1}{h_p^z}\int_{z'_{p-1}}^{z'_p} \frac{\mathrm{d}z'}{\lambda'(r'_i,z',\bar{\bar{u}})}\right]^{-1} \tag{30}$$

and $\bar{\bar{u}}$ is the linear fulfilment of the difference function $y_{i,p}^{j+1/2}$ at fixed i.

To approximate the integrals in formulae (28) and (30) we use the trapezoidal rule taking into account the discontinuities of the coefficients not only on interior fixed boundaries of the domain D', but also on the moving lines of discontinuity $u=1$ and $u=1+\eta$. The systems of linear algebraic equations (27) and (29) we solve by Thomas's procedure.

4. Numerical results and interpretation

The formulated problem is characterized by a lot of dimensionless parameters. That is why a basic variant is chosen to find the principal regularities. With respect to it the effect of the variable parameters of the problem is investigated. The calculation of the solidification of an alloy in a homogeneous metal mould (the thermophysical characteristics of the bush are equal to the thermophysical characteristics of the basic part of the mould) when the gas gapping and the mould paint between the ingot and the mould are not taken into account and the upper surface of the ingot and the mould are heat insulated is chosen as a basic variant. The dimensionless parameters in the basic variant have the following values; $R'_n=1.165$, $R'_k=1.66$, $H'_k=0.16$, $H'_n=2.4$, $H_m=3$, $c'_1=1$, $c'_k=c'_n=1.25$, $\lambda'_1=1$, $\lambda'_k=\lambda'_n=2$, $Ste=0.5$, $\eta=0.05$, $a=0.2$, $d'_{s_0}=1$, $\tilde{u}_0=1.07$, $u_\infty=0.01$, $Bi_1=Bi_2=0.2$, $Bi_k=100$, $Bi_3=Bi_g=Bi_v=Bi_{\sigma_1}=Bi_{\sigma_2}=Ki=t'_q=0$.

At the analysis of the solidification of a molten metal in a metal mould the following characteristics of the process are very important for the casting technology: the time of the complete crystallization of the ingot, the form and the speed of the movement of isotherm $u=u_s=1$ and the maximal temperature the metal mould interface reaches. To obtain the directional crystallization in vertical ingots it is necessary to realize conditions at which the front of solidification is shifted from bottom to top. The maximal temperature of the metal mould interface determines the rate of its wearing out.

In the circumstances of heat insulated upper surface of the ingot and the metal mould ($Bi_3=Ki=Bi_{\sigma_1}=0$) in a large interval of geometrical (H'_k, H'_n, H'_m, R'_k, R'_n) and physical parameters the time of the crystallization of cylindrical ingots is determined by the time of the crystallization of an infinitely extended ingot along z'. In the context of such one-dimensional by space

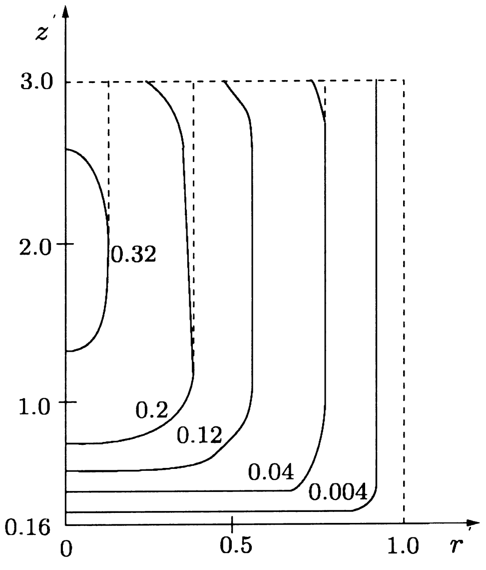

Fig. 2. $Bi_3=0.2$ (in the basic variant $Bi_3=0$).

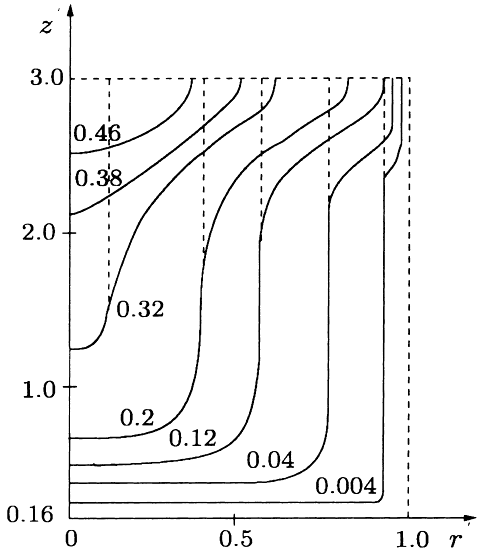

Fig. 3. $c_n'=0.6$, $\lambda_n'=0.06$ (there are not any exothermal mixtures).

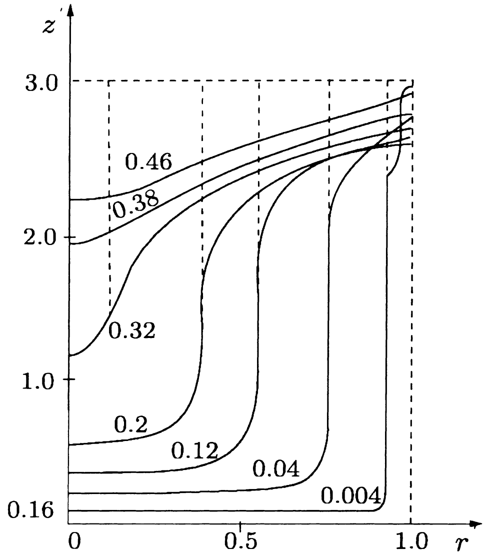

Fig. 5. $Ki=2.5$, $t_q'=0.15$, $c_n'=0.6$, $\lambda_n'=0.06$.

approach the effects of the variable parameters on the solidification are considered in detail. Some results of this research are published in [11]. In this connection we note only that good contrivances for controlling of crystallization are the parameters $\tilde{u}_0$, λ_k', c_k', R_k' and a particularly powerful contrivance is the parameter Bi_v. Moreover, in almost all cases the decrease of the radial velocity of crystallization is inevitably associated with increase of the attained maximal temperature of the metal mould. Exception in this respect is parameter Bi_v which when appreciably slowing down the process of crystallization decreases this temperature.

For infinitely extended ingots a deep pool is formed, where the appearance of defects is possible. To prevent these defects the efficient manner is the control of the

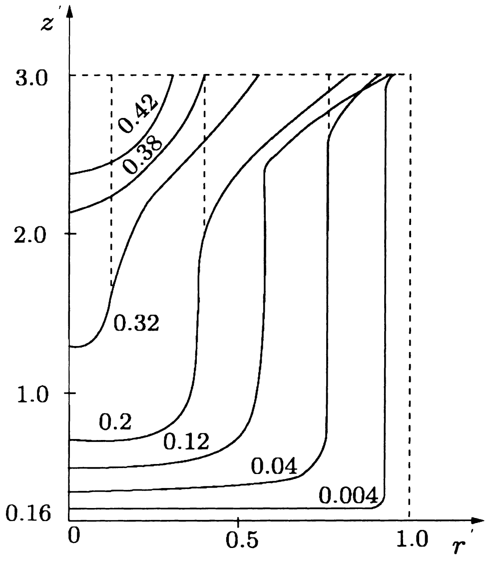

Fig. 4. $Ki=2.5$, $t_q'=0.15$ (there is not a bush).

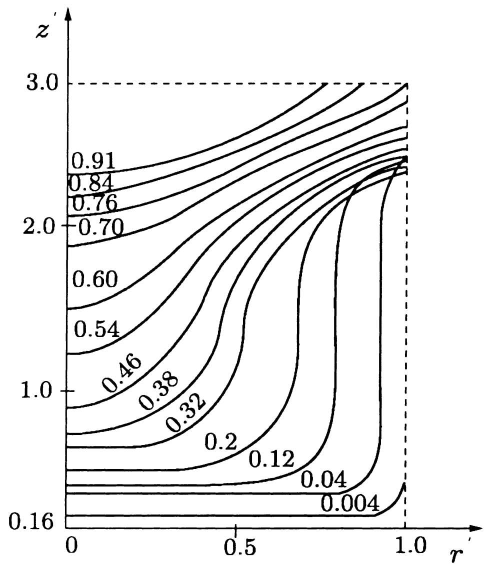

Fig. 6.

flows on the upper surface of the ingot and the metal mould. Such a problem is essentially two-dimensional. We shall mentioned some specific examples of its numerical solution.

In Fig. 2 the isotherm $u = u_s = 1$ at $Bi_3 = 0.2$ at variable moments of time are presented. The other parameters stay the same as in the basic variant. From this figure we notice that inside the ingot, an area of liquid metal stays. Isotherms $u = 1$ at $Bi_3 = 0$ are shown by a dotted line.

We have done a lot of numerical experiments in order to examine the effects of the thermophysical characteristics and the sizes of the bush and also the exothermal mixtures and the time of their acting. In Fig. 3 the effect of the bush is presented. Here $c_n' = 0.6$, $\lambda_n' = 0.06$ is set. The effect of the exothermic mixes at $Ki = 2.5$, $t_q = 0.15$ is shown in Fig. 4. Investigations of the combined action of these two factors are also carried out. In Fig. 5 at $Ki = 2.5$, $t_q = 0.15$, $c_n' = 0.6$, $\lambda_n' = 0.06$ the combined effect of the bush and the exothermal mixtures on the alteration of the depth and especially on the form of the pool, may be traced. Summarizing the result obtained it may be said, that the bush and the exothermal mixtures are particularly powerful control factors of the process in its upper part.

The movement of the front of the solidification when the effect of all the factors is taken into account—bush ($c_n' = 0.6$, $\lambda_n' = 0.06$), exothermal mixtures ($Ki = 2.5$, $t_q = 0.15$), gas gapping ($Bi_g = 0.067$, $Bi_{\sigma_2} = 0.9$, $d_{s_0}' = 0.13$) and mould paint ($Bi_v = 0.003$) is shown in Fig. 6.

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